Quadratics (2)

1. **(a)** Determine the restrictions for λ such that the equation $x^2 - 6x - 1 + \lambda(2x+1) = 0$ has real roots.

(b) Determine the restrictions for λ such that the equation $\lambda x^2 - 6x - 1 + \lambda(x+1) = 0$ has real roots.

(a) $x^2 - 6x - 1 + \lambda(2x+1) = 0 \iff x^2 + (2\lambda - 6)x + (\lambda - 1) = 0$ has real roots if and only if

$\Delta = (2\lambda - 6)^2 - 4(\lambda - 1) \geq 0$

$4\lambda^2 - 28\lambda + 40 \geq 0$

$\lambda^2 - 7\lambda + 10 \geq 0$

$(\lambda - 2)(\lambda - 5) \geq 0$

$\begin{cases} \lambda - 2 \leq 0 \\ \lambda - 5 \leq 0 \end{cases}$ or $\begin{cases} \lambda - 2 \geq 0 \\ \lambda - 5 \geq 0 \end{cases}$

$\therefore \lambda \leq 2$ or $\lambda \geq 5$

(b) $\lambda x^2 - 6x - 1 + \lambda(x+1) = 0 \iff \lambda x^2 + (\lambda - 6)x + (\lambda - 1) = 0$ has real roots if and only if

$\Delta = (\lambda - 6)^2 - 4\lambda(\lambda - 1) \geq 0$

$36 - 8\lambda - 3\lambda^2 \geq 0$

$3\lambda^2 + 8\lambda - 36 \leq 0$

$\left(\lambda - \frac{-4-2\sqrt{31}}{3}\right)\left(\lambda - \frac{-4+2\sqrt{31}}{3}\right) \leq 0$

$\therefore \frac{-4-2\sqrt{31}}{3} \leq \lambda \leq \frac{-4+2\sqrt{31}}{3}$

(Check: If $\lambda = 0$, the given equation is reduced to $-6x - 1 = 0$, it is no longer quadratic but has real root $x = -\frac{1}{6}$.)

2. **(a)** If α and β are the roots of the equation $ax^2 + 2bx + c = 0$, where a, b, c are real numbers and $a \neq 0$, show that $\alpha + \beta = -\frac{2b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(b) If the above equation has real roots and m, n are real numbers such that $m^2 > n > 0$, show that the equation $ax^2 + 2mbx + nc = 0$ also has real solutions.

(a) Since α and β are the roots of the equation $ax^2 + 2bx + c = 0 \quad \ldots(1)$,

therefore

$$a(x - \alpha)(x - \beta) = 0$$

or

$$ax^2 + a(\alpha + \beta) + a\alpha\beta = 0 \quad \ldots(2)$$

Compare coefficients of (1), (2), $a(\alpha + \beta) = 2b,\ a\alpha\beta = c$

$$\therefore\ \alpha+\beta=-\frac{2b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

(b) If the equation $ax^2+2bx+c=0$ has real roots

$$\Delta=(2b)^2-4ac\geq 0$$

Therefore $b^2-ac\geq 0 \Longrightarrow b^2\geq ac \quad \ldots.(3)$.

For the equation $ax^2+2mbx+nc=0$,

$$\Delta=(2mb)^2-4nc=4m^2b^2-4nac$$

Case 1, If $ac<0$, then $-4nac>0$, (given $n>0$)

$$\therefore \Delta=4m^2b^2-4nac>0 \quad (m^2b^2\geq 0)$$

Case 2, If $ac\geq 0$, from (3), $b^2\geq ac\geq 0$ and multiply by $m^2\geq 0$,

$$4m^2b^2\geq 4m^2ac>4nac\geq 0$$

$$\therefore \Delta=4m^2b^2-4nac>0$$

In both cases, $\Delta\geq 0$, therefore the equation $ax^2+2mbx+nc=0$ also has real solutions.

3. If α and β are the roots for the equation $px^2+qx+r=0$, where p,q,r are real numbers and $p,q\neq 0$, show that $\alpha+\beta=-\frac{q}{p}$, $\alpha\beta=\frac{r}{p}$

If γ and δ are the roots of $qx^2+rx+p=0$, show that

(a) $(\alpha-\gamma)(\alpha-\delta)=\frac{q\alpha^2+r\alpha+p}{q}$

(b) $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)=\frac{p^3+q^3+r^3-3pqr}{pq^2}$

(c) Hence or otherwise, determine the relationship needed for the above two quadratic equations to have one **common** real root.

$$px^2+qx+r\equiv p(x-\alpha)(x-\beta)=p[x^2-(\alpha+\beta)x+\alpha\beta]$$

$$\therefore -p(\alpha+\beta)=q,\ p\alpha\beta=r \iff \alpha+\beta=-\frac{q}{p}\ ,\ \alpha\beta=\frac{r}{p}$$

(a) By Vieta's Theorem, $\gamma+\delta=-\frac{r}{q}$, $\gamma\delta=\frac{p}{q}$

$$(\alpha-\gamma)(\alpha-\delta)=\alpha^2-(\gamma+\delta)\alpha+(\gamma\delta)=\alpha^2-\left(-\frac{r}{q}\right)\alpha+\left(\frac{p}{q}\right)=\frac{q\alpha^2+r\alpha+p}{q}$$

(b) $(\alpha-\gamma)(\alpha-\delta)=\frac{q\alpha^2+r\alpha+p}{q}$

Similarly in **(a)**, we have $(\beta-\gamma)(\beta-\delta)=\frac{q\beta^2+r\beta+p}{q}$

$$(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)=\left(\frac{q\alpha^2+r\alpha+p}{q}\right)\left(\frac{q\beta^2+r\beta+p}{q}\right)$$

$$=\frac{q^2\,\alpha^2\,\beta^2+q\,r\,\beta\,\alpha^2+q\,r\,\alpha\,\beta^2+p\,q\,\alpha^2+p\,q\,\beta^2+p\,r\,\alpha+p\,r\,\beta+\alpha\,\beta\,r^2+p^2}{q^2}$$

$$=\frac{q^2\,(\alpha\beta)^2+q\,r\alpha\beta\,(\alpha+\beta)+p\,q\,(\alpha^2+\beta^2)+p\,r\,(\alpha+\beta)+\alpha\,\beta\,r^2+p^2}{q^2}$$

$$=\frac{q^2\left(\frac{r}{p}\right)^2+q\,r\left(\frac{r}{p}\right)\left(-\frac{q}{p}\right)+p\,q\left[\left(-\frac{q}{p}\right)^2-2\left(\frac{r}{p}\right)\right]+p\,r\left(-\frac{q}{p}\right)+\left(\frac{r}{p}\right)r^2+p^2}{q^2}$$

$$=\frac{\frac{q^2r^2}{p^2}-\frac{q^2r^2}{p^2}+\frac{q^3}{p}-2qr-qr+\frac{r^3}{p}+p^2}{q^2}=\frac{p^2+\frac{q^3}{p}+\frac{r^3}{p}-3qr}{q^2}=\frac{p^3+q^3+r^3-3pqr}{pq^2}$$

(c) The above two equations to have one common real root

$\Leftrightarrow (\alpha=\gamma)$ or $(\alpha=\delta)$ or $(\beta=\gamma)$ or $(\beta=\delta) \Leftrightarrow (\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)=0$

$\Leftrightarrow \frac{p^3+q^3+r^3-3pqr}{pq^2}=0$

$\Leftrightarrow p^3+q^3+r^3-3pqr=0$ and $p,q\neq 0$

$\Leftrightarrow (p+q+r)(p^2+q^2+r^2-pq-qr-rq)=0$ and $p,q\neq 0$

$\Leftrightarrow \frac{1}{2}(p+q+r)[(p-q)^2+(q-r)^2+(r-p)^2]=0$ and $p,q\neq 0$

$\Leftrightarrow p+q+r=0$ and $p,q\neq 0$

Note that $(p-q)^2+(q-r)^2+(r-p)^2=0 \Leftrightarrow p=q=r=0$ should be rejected since $p,q\neq 0$.

In order that "The above two equations to have one common **real** root.", we need :

1. For $px^2+qx+r=0$, $\Delta=q^2-4pr\geq 0$
2. For $qx^2+rx+p=0$, $\Delta=r^2-4pq\geq 0$

In conclusion,

The above two equations to have one common real root

$\Leftrightarrow p+q+r=0$ and $q^2\geq 4pr$ and $r^2\geq 4pq$ and $p,q\neq 0$

4. If $ax^2+2bx+c=0\ (a\neq 0)$ and $y=x+\frac{1}{x}$, prove that

$$acy^2+2b(c+a)y+(a-c)^2+4b^2=0$$

Hence, if α and β are the roots of the equation $ax^2+2bx+c=0$, show that

$$\left(\alpha+\frac{1}{\alpha}\right)^2+\left(\beta+\frac{1}{\beta}\right)^2=\frac{4b^2(a^2+c^2)-2ac(a-c)^2}{a^2c^2}$$

Method 1

Since α and β are the roots of the equation $ax^2 + 2bx + c = 0$

$$\alpha + \beta = -\frac{2b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\left(\alpha + \frac{1}{\alpha}\right) + \left(\beta + \frac{1}{\beta}\right) = \alpha + \beta + \frac{\alpha+\beta}{\alpha\beta} = -\frac{2b}{a} + \frac{-\frac{2b}{a}}{\frac{c}{a}} = -\frac{2b}{a} - \frac{2b}{c} = -\frac{2b(c+a)}{ac}$$

$$\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right) = \alpha\beta + \frac{1}{\alpha\beta} + \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{c}{a} + \frac{1}{\frac{c}{a}} + \frac{\left(-\frac{2b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{\frac{c}{a}} = \frac{c}{a} + \frac{a}{c} + \frac{4b^2-2ac}{ac} = \frac{a^2+c^2}{ac} + \frac{4b^2-2ac}{ac}$$

$$= \frac{(a-c)^2+4b^2}{ac}$$

The equation with roots $\left(\alpha + \frac{1}{\alpha}\right)$ and $\left(\beta + \frac{1}{\beta}\right)$ is

$$y^2 + \frac{2b(c+a)}{ac}y + \frac{(a-c)^2 + 4b^2}{ac} = 0$$

or $$acy^2 + 2b(c+a)y + (a-c)^2 + 4b^2 = 0$$

Method 2

$$y = x + \frac{1}{x} = \frac{x^2+1}{x} = \frac{ax^2+a}{ax} = \frac{(ax^2+2bx+c) - 2bx + (a-c)}{ax} = \frac{-2bx + (a-c)}{ax}$$

$$axy = -2bx + (a-c)$$

$$(ay+2b)x = (a-c)$$

$$x = \frac{a-c}{ay+2b}$$

Since $ax^2 + 2bx + c = 0$

$$a\left(\frac{a-c}{ay+2b}\right)^2 + 2b\left(\frac{a-c}{ay+2b}\right) + c = 0$$

$$c(ay+2b)^2 + 2b(a-c)(ay+2b) + a(a-c)^2 = 0$$

$$c\,a^2\,y^2 + 2\,a\,b\,c\,y + 2\,b\,y\,a^2 + a\,c^2 + 4\,a\,b^2 - 2\,c\,a^2 + a^3 = 0$$

$$ac\,y^2 + 2\,b\,c\,y + 2\,ab\,y + c^2 + 4\,b^2 - 2\,ac + a^2 = 0$$

$$acy^2 + 2b(c+a)y + (a-c)^2 + 4b^2 = 0$$

By Vieta Theorem,

$\left(\alpha + \frac{1}{\alpha}\right) + \left(\beta + \frac{1}{\beta}\right) = -\frac{2b(c+a)}{ac}$

$\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right) = \frac{(a-c)^2+4b^2}{ac}$

Hence $\left(\alpha + \frac{1}{\alpha}\right)^2 + \left(\beta + \frac{1}{\beta}\right)^2 = \left[\left(\alpha + \frac{1}{\alpha}\right) + \left(\beta + \frac{1}{\beta}\right)\right]^2 - 2\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right) = \left[-\frac{2b(c+a)}{ac}\right]^2 - \frac{(a-c)^2+4b^2}{ac}$

$= \frac{4b^2(a^2+c^2)-2ac(a-c)^2}{a^2c^2}$

5. If x_1 and x_2 are the roots of the equation $x^2 - (a+d)x + ad - bc = 0$.
Prove that the equation with roots $x_1{}^3$ and $x_2{}^3$ is

$$x^2 - (a^3 + d^3 + 3abc + 3bcd)x + (ad - bc)^3 = 0$$

Method 1

$x_1 + x_2 = a + d$

$x_1x_2 = ad - bc$

Hence, $x_1{}^3 + x_2{}^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2) = (a + d)^3 - 3(ad - bc)(a + d)$

$= a^3 + d^3 + 3abc + 3bcd$

$x_1{}^3x_2{}^3 = (ad - bc)^3$

Therefore the equation with roots $x_1{}^3$ and $x_2{}^3$ is

$$x^2 - (a^3 + d^3 + 3abc + 3bcd)x + (ad - bc)^3 = 0$$

Method 2

Use the transformation $y = x^3$ or $x = \sqrt[3]{y}$ and put it in the given quadratic equation

The required equation is therefore $\sqrt[3]{y}^2 - (a+d)\sqrt[3]{y} + ad - bc = 0$

$\sqrt[3]{y}[\sqrt[3]{y} - (a+d)] = -(ad - bc)$

$$y\left[y - 3(a+d)\sqrt[3]{y}^2 + 3(a+d)^2\sqrt[3]{y} - (a+d)^3\right] = -(ad - bc)^3$$

$$y\left\{y - 3(a+d)\left[\sqrt[3]{y}^2 - (a+d)\sqrt[3]{y} + ad - bc\right] + 3(a+d)(ad-bc) - (a+d)^3\right\} + (ad-bc)^3 = 0$$

$y\{y - 3(a+d)[0] + 3(a+d)(ad-bc) - (a+d)^3\} + (ad-bc)^3 = 0$

$y^2 - (a^3 + d^3 + 3abc + 3bcd)y + (ad - bc)^3 = 0$

6. If α and β are the roots for the equation $x^2 + bx + c = 0$ and γ and δ are the roots for the equation $x^2 + \lambda bx + \lambda^2c = 0$ where b, c and λ are real numbers, show that the equations with roots $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ is $x^2 - \lambda b^2x + 2\lambda^2c(b^2 - 2c) = 0$ and show that the roots of this new equation are real.

By Vieta Theorem, $\alpha + \beta = -b$, $\alpha\beta = c$

and $\gamma + \delta = -\lambda b$, $\gamma\delta = \lambda^2 c$

New sum of roots $= (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma) = (\alpha + \beta)(\gamma + \delta) = (-b)(-\lambda b) = \lambda b^2$

New product of roots $= (\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma) = \alpha^2\gamma\delta + \alpha\beta\gamma^2 + \alpha\beta\delta^2 + \gamma\delta\beta^2$

$= \alpha\beta(\gamma^2 + \delta^2) + \gamma\delta(\alpha^2 + \beta^2)$

$= \alpha\beta[(\gamma + \delta)^2 - 2\gamma\delta] + \gamma\delta[(\alpha + \beta)^2 - 2\alpha\beta]$

$= c[(-\lambda b)^2 - 2\lambda^2 c] + \lambda^2 c[(-b)^2 - 2c]$

$= \lambda^2 c(b^2 - 2c) + \lambda^2 c(b^2 - 2c) = 2\lambda^2 c(b^2 - 2c)$

The new equation is therefore: $x^2 - \lambda b^2 x + 2\lambda^2 c(b^2 - 2c) = 0$

$$\Delta = (-\lambda b^2)^2 - 4[2\lambda^2 c(b^2 - 2c)] = \lambda^2 b^4 + c\lambda^2(16c - 8b^2)$$

$$= \lambda^2[b^4 - 8b^2c + 16c^2] = \lambda^2[b^4 - 8b^2c + 16c^2]$$

$$= \lambda^2[b^2 - 4c]^2 \geq 0$$

Therefore the roots of this new equation are real.

(If $\lambda = 0$ or $b = \pm 2\sqrt{c}$ the new equation has equal roots)

7. If α and β are the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ form the equation where the roots are $\frac{1}{\alpha - 4\beta}$ and $\frac{1}{\beta - 4\alpha}$.

Method 1

Since α and β are the roots of the equation $ax^2 + bx + c = 0$, $(a \neq 0)$

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

New sum of roots = $\frac{1}{\alpha-4\beta} + \frac{1}{\beta-4\alpha} = \frac{(\alpha-4\beta)+(\beta-4\alpha)}{(\alpha-4\beta)(\beta-4\alpha)} = \frac{-3\alpha-3\beta}{17\alpha\beta-4\alpha^2-4\beta^2} = \frac{-3(\alpha+\beta)}{17\alpha\beta-4[(\alpha+\beta)^2-2\alpha\beta]}$

$$= \frac{-3\left(-\frac{b}{a}\right)}{17\left(\frac{c}{a}\right)-4\left[\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)\right]} = \frac{3ab}{17ac-4[b^2-2ac]} = \frac{3ab}{25ac-4b^2}$$

New product of roots = $\left(\frac{1}{\alpha-4\beta}\right)\left(\frac{1}{\beta-4\alpha}\right) = \frac{1}{17\alpha\beta-4\alpha^2-4\beta^2} = \frac{1}{17\alpha\beta-4[(\alpha+\beta)^2-2\alpha\beta]}$

$$= \frac{1}{17\left(\frac{c}{a}\right)-4\left[\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)\right]} = \frac{a^2}{25ac-4b^2}$$

Therefore the equation where the roots are $\frac{1}{\alpha-4\beta}$ and $\frac{1}{\beta-4\alpha}$ is

$$x^2 - \frac{3ab}{25ac-4b^2}x + \frac{1}{25ac-4b^2} = 0$$

or $(25ac - 4b^2)x^2 - 3abx + a^2 = 0$

Method 2

Since α and β are the roots of the equation $ax^2 + bx + c = 0$, $(a \neq 0)$

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\frac{1}{\alpha-4\beta} = \frac{1}{(\alpha+\beta)-5\beta} = \frac{1}{-\frac{b}{a}-5\beta} = \frac{a}{-b-5a\beta}$$

Similarly $\frac{1}{\beta-4\alpha} = \frac{a}{-b-5a\alpha}$

Therefore we use the transformation : $y = \frac{a}{-b-5ax}$ or $x = -\frac{by+a}{5ay}$.

Therefore the equation where the roots are $\frac{1}{\alpha-4\beta}$ and $\frac{1}{\beta-4\alpha}$ is

$$a\left(-\frac{by+a}{5ay}\right)^2 + b\left(-\frac{by+a}{5ay}\right) + c = 0$$

$$a(by + a)^2 - b(by + a)(5ay) + c(5ay)^2 = 0$$

$$(by + a)^2 - b(by + a)(5y) + ac(5y)^2 = 0$$

$$(25ac - 4b^2)y^2 - 3aby + a^2 = 0 \quad \text{(where y is the variable)}$$

8. If one of the roots for the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$ is the positive square root of the other, show that $b^3 = ac(3b - a - c)$

Hence, find the value(s) of y such that the roots of the quadratic equation

$$27x^2 + 6x - (y + 2) = 0$$

has a root that is the positive square root of the other.

Let $\alpha, \sqrt{\alpha}$, $(\alpha > 0)$ be the roots of the quadratic equation $ax^2 + bx + c = 0$, $(a \neq 0)$.

By Vieta Theorem, $\alpha + \sqrt{\alpha} = -\frac{b}{a}$ ... (1)

$\alpha\sqrt{\alpha} = \frac{c}{a}$ ... (2)

$(1)^3$, $\alpha^3 + 3\alpha^2\sqrt{\alpha} + 3\alpha(\sqrt{\alpha})^2 + (\sqrt{\alpha})^3 = -\frac{b^3}{a^3}$

$$(\alpha\sqrt{\alpha})^2 + 3\alpha(\alpha\sqrt{\alpha}) + 3\sqrt{\alpha}(\alpha\sqrt{\alpha}) + \alpha\sqrt{\alpha} = -\frac{b^3}{a^3}$$

$$\frac{c^2}{a^2} + 3\alpha\left(\frac{c}{a}\right) + 3\sqrt{\alpha}\left(\frac{c}{a}\right) + \left(\frac{c}{a}\right) = -\frac{b^3}{a^3}$$

$$\frac{c^2}{a^2} + 3\left(\frac{c}{a}\right)(\alpha + \sqrt{\alpha}) + \left(\frac{c}{a}\right) = -\frac{b^3}{a^3}$$

$$\frac{c^2}{a^2} + 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right) + \left(\frac{c}{a}\right) = -\frac{b^3}{a^3}$$

$$ac^2 - 3abc + a^2c = -b^3$$

$$b^3 = ac(3b - a - c)$$

Since $27x^2 + 6x - (y+2) = 0$ has a root that is the square root of the other, from $b^3 = ac(3b - a - c)$

$6^3 = 27[-(y+2)](3 \times 6 - 27 + (y+2))$

$27y^2 - 135y - 162 = 0$

$y^2 - 5y - 6 = 0$

$y = -1$ or $y = 6$

Check : **(1)** If $y = -1$, $27x^2 + 6x - (y+2) = 0$ becomes $27x^2 + 6x - 1 = 0$, then $x = -\frac{1}{3}$ or $x = \frac{1}{9}$

(2) If $y = 6$, $27x^2 + 6x - (y+2) = 0$ becomes $27x^2 + 6x - 8 = 0$, then $x = -\frac{2}{3}$ or $x = \frac{4}{9}$

Since in both cases, one root is not the **positive** square root of the other, there is **no solution** for y .

9. If α is a root of the equation $x^5 - 1 = 0$, where $\alpha \neq 1$, form a quadratic equation with roots $\alpha^4 + \alpha$ and $\alpha^3 + \alpha^2$.

$(\alpha^4 + \alpha) + (\alpha^3 + \alpha^2) = \alpha^4 + \alpha^3 + \alpha^2 + \alpha = (\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) - 1 = \frac{\alpha^5 - 1}{\alpha - 1} - 1 = \frac{0}{\alpha - 1} - 1 = -1$

$(\alpha^4 + \alpha)(\alpha^3 + \alpha^2) = \alpha^7 + \alpha^6 + \alpha^4 + \alpha^3 = \alpha^5\alpha^2 + \alpha^5\alpha + \alpha^4 + \alpha^3 = (1)\alpha^2 + (1)\alpha + \alpha^4 + \alpha^3$

$= \alpha^4 + \alpha^3 + \alpha^2 + \alpha = (\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) - 1 = \frac{\alpha^5 - 1}{\alpha - 1} - 1 = \frac{0}{\alpha - 1} - 1 = -1$

Hence the quadratic equation with roots $\alpha^4 + \alpha$ and $\alpha^3 + \alpha^2$ is

$x^2 + x - 1 = 0$

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